Advanced Quantum Field Theory (2024/2025)
TP 5 - The Gross–Neveu model

The Gross–Neveu model

The Gross–Neveu model[1] was developed as a toy model for QCD. It exhibits asymptotic freedom and dynamical symmetry breaking due to quantum effects. It furthermore has a "flavor" parameter N and allows for an exact treatment in the $N \to \infty$ limit. We will explore these features in the following exercise.

The Gross–Neveu model is a model of N fermions ψ_i, $i = 1, \dots N$ in two spacetime dimensions defined by[2]

$$\mathcal{L} = \bar{\psi}_i i\partial\!\!\!/ \psi_i + \frac{1}{2} g^2 (\bar{\psi}_i \psi)^2 \,. \tag{1}$$

The two gamma matrices γ^0, γ^1 can be taken as

$$\gamma^0 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \gamma^1 = i\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2}$$

Define furthermore the matrix

$$\gamma^5 = \gamma^0 \gamma^1 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

These matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \qquad \{\gamma^\mu, \gamma^5\} = 0, \qquad (\gamma^5)^\dagger = \gamma^5 \,. \tag{4}$$

1. Show that this theory is invariant under the $\mathbb{Z}_2$ symmetry

$$\psi_i \to \gamma^5 \psi \,. \tag{5}$$

   Note that this symmetry forbids a fermion mass term.

2. Show that this theory is power-counting renormalizable in two dimensions.

3. Show that the Lagrangian (1) can be equivalently written as

$$\mathcal{L}' = \bar{\psi}_i i\partial\!\!\!/ \psi_i - \sigma \bar{\psi}_i \psi_i - \frac{1}{2g^2}\sigma^2, \tag{6}$$

   where σ is a non-dynamical auxiliary field.[3] We want to use dimensional regularization later on. Introduce a dimensionful parameter μ such that the coupling constant is dimensionless and ψ_i has canonical weight in d dimensions.

[1] D. J. Gross and A. Neveu, "Dynamical Symmetry Breaking in Asymptotically Free Field Theories," Phys. Rev. D **10** (1974), 3235

[2] In this exercise, we will work in $(+-)$ signature.

[3] In this and related context, this trick is known as a *Hubbard-Stratonovich transformation*.

4. Consider now the functional integral

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma \exp\left[i\int d^{2-2\epsilon}x\left(\bar{\psi}_i i\partial\!\!\!/\psi_i - \mu^\epsilon\sigma\bar{\psi}_i\psi_i - \frac{1}{2g^2}\sigma^2\right)\right]. \tag{7}$$

Note that the integral over the fermionic fields is Gaussian. Determine the effective action of σ by integrating over the fermionic fields. You should find

$$\Gamma[\sigma] = \frac{N}{i}\mathrm{Tr}\ln(i\partial\!\!\!/ - \mu^\epsilon\sigma(x)) + N\int d^2x\left(-\frac{1}{2g^2N}\sigma^2\right). \tag{8}$$

5. We will be interested in the effective potential of σ so that we can restrict to constant σ. Evaluate the determinant for constant $\sigma = \sigma_0$ in dimensional regularization. Compute the trace of the operator by going to momentum space. There will still be a trace over spinor indices left. Evaluate this to find

$$\mathrm{Tr}\ln(i\partial\!\!\!/ - \mu^\epsilon\sigma_0) = \int d^dx\int\frac{d^dk}{(2\pi)^d}\ln(-k^2 + \mu^{2\epsilon}\sigma_0^2). \tag{9}$$

Use dimensional regularization to evaluate the integral and use minimal modified subtraction to renormalize the coupling constant, i.e., absorb the infinite term plus the constant pieces $-\gamma_E$ and $\ln 4\pi$ in the coupling constant. You should find the effective potential

$$V_{eff} = \frac{\sigma^2}{4\pi}\left(\ln\left(\frac{\sigma^2}{\mu^2}\right) - 1\right) + \frac{\sigma^2}{2g^2N}, \tag{10}$$

where μ is a scale introduced to make g dimensionless.

6. Compute the minimum of the potential and compare it with the result coming from the classical potential (6). We find that the field σ acquires a vacuum expectation value which breaks the symmetry.

7. Compute the beta function of the Gross–Neveu model using the fact that the effective potential is independent of the renormalization scale μ. You should find

$$\beta(g) = -\frac{Ng^3}{2\pi}. \tag{11}$$

We therefore conclude that the theory is asymptotically free.

8. Note that the effective potential only depends on the combination $\lambda = g^2N$, often called *'t Hooft coupling*. Argue that the limit $N \to \infty$ with λ fixed corresponds to the classical limit $\hbar \to 0$.

Useful formulas

Functional determinant for fermions

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left[i\int d^2x(\bar{\psi}(i\partial\!\!\!/ - m)\psi)\right] = \det\big(i\partial\!\!\!/ - m\big). \tag{12}$$

Matrix logarithm

$$\mathrm{Tr}\ln A = \ln\det A. \tag{13}$$

Dimensional regularization of a divergent integral:

$$\int\frac{d^dk}{(2\pi)^d}\log(-k^2+m^2) = i\int\frac{d^dk_E}{(2\pi)^d}\log(k_E^2+m^2) = -i\partial_\alpha\int\frac{d^dk_E}{(2\pi)^d}\frac{1}{(k_E^2+m^2)^\alpha}|_{\alpha=0} = -i\frac{(m^2)^{d/2}}{(4\pi)^{d/2}}\Gamma(-d/2). \tag{14}$$